Q1a

$$x = \int \frac{dx}{dt}\, dt$$

a) $5 - \sin 2t + \frac{dx}{dt} = 0$

$$\frac{dx}{dt} = \sin 2t - 5$$

$$x = \int (\sin 2t - 5)\, dt$$

$$\boxed{= -\frac{1}{2}\cos 2t - 5t + c}$$

$\int \sin 2t \, dt = -\frac{1}{2}\cos 2t + c$

(reverse chain rule)

Q1b

$$V = \int \frac{dV}{dx}\, dx$$

b) $3e^{4x} - \frac{dV}{dx} = 2$

$$\frac{dV}{dx} = 3e^{4x} - 2$$

$$V = \int (3e^{4x} - 2)\, dx$$

$\int 3e^{4x} dx = \frac{3}{4}e^{4x} + c$ (reverse chain rule)

$$V = \frac{3}{4}e^{4x} - 2x + c \quad \Big\}\ \text{general solution}$$

But when $x = 0$, $V = -4$, so

$$-4 = \frac{3}{4}e^{0} - 2(0) + c$$

$$-4 = \frac{3}{4} + c$$

$$c = -4 - \frac{3}{4} = -\frac{19}{4}$$

Use boundary condition to find value of c

$$\boxed{V = \frac{3}{4}e^{4x} - 2x - \frac{19}{4}} \quad \Big\}\ \text{particular solution}$$

Q2a

(a) Show that the general solution to the differential equation

$$\frac{dy}{dx} = 3x^2 y, \qquad y \neq 0$$

is

$$y = Ae^{x^3}$$

where A is a constant.

[5]

(b) On the same set of axes sketch the graphs of the solution for the instances where
(i) the constant A is greater than 0
(ii) the constant A is less than 0
In each case be sure to state where the graph intercepts the y-axis.

[3]

a)

$$\frac{dy}{dx} = 3x^2 y$$

$$\int \frac{1}{y}\, dy = \int 3x^2\, dx$$

$$\ln|y| = x^3 + c$$

$e^{\ln y} = y$

If $y > 0$, then $|y| = y$ and

$$\ln y = x^3 + c$$

$$y = e^{x^3 + c} = (e^{x^3})(e^{c})$$

$$y = Ae^{x^3} \quad (\text{where } A = e^{c})$$

$e^{\ln(-y)} = -y$

If $y < 0$, then $|y| = -y$ and

$$\ln(-y) = x^3 + c$$

$$-y = e^{x^3 + c} = (e^{x^3})(e^{c})$$

$$y = -(e^{c})(e^{x^3})$$

$$y = Ae^{x^3} \quad (\text{where } A = -e^{c})$$

Separation of Variables

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

Q2b

(a) Show that the general solution to the differential equation

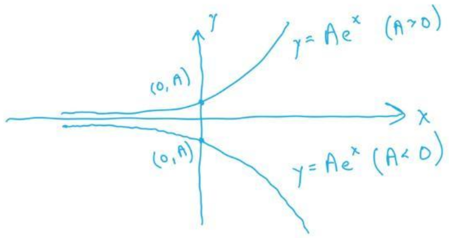
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 y, \qquad y \neq 0$$

is

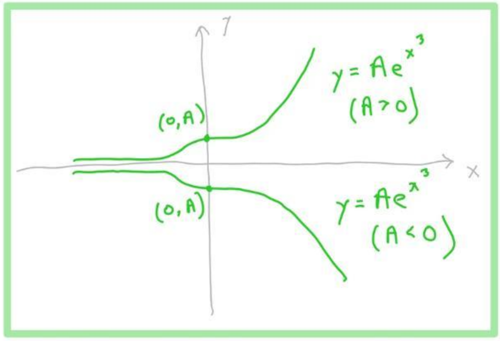
$$y = Ae^{x^3}$$

where A is a constant.

(b) On the same set of axes sketch the graphs of the solution for the instances where
(i) the constant A is greater than 0
(ii) the constant A is less than 0
In each case be sure to state where the graph intercepts the y-axis.



b)



Ae^{x^3} is similar to Ae^x, although it increases faster when x is positive and approaches zero faster when x is negative. The biggest difference in the general shape is that Ae^{x^3} 'levels out' a bit between $x = -1$ and $x = 1$. This is because
$-1 < x < 0 \implies x < x^3 < 0$
and $0 < x < 1 \implies 0 < x^3 < x$

Q3

The acceleration of a particle is given by the differential equation

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = 18t - 4$$

where s is the displacement of the particle in metres from a fixed reference point O, and t is the time in seconds.

The particle starts its journey at point O at $t = 0$, and has an initial velocity of 1 m s^{-1}.

Find an expression for the displacement of the particle in terms of t.

[4

$$\frac{ds}{dt} = \int \frac{d^2s}{dt^2}\, dt \qquad s = \int \frac{ds}{dt}\, dt$$

$$v = \frac{ds}{dt} = \int (18t - 4)\, dt$$
$$= 9t^2 - 4t + c_1$$

But when $t = 0$, $v = 1$, so
$$1 = 9(0)^2 - 4(0) + c_1 \implies c_1 = 1$$
$$v = 9t^2 - 4t + 1$$

$$s = \int (9t^2 - 4t + 1)\, dt$$
$$= 3t^3 - 2t^2 + t + c_2$$

But when $t = 0$, $s = 0$, so
$$0 = 3(0)^3 - 2(0)^2 + 0 + c_2 \implies c_2 = 0$$

$$\boxed{s = 3t^3 - 2t^2 + t}$$

Q4a

A large container of water is leaking at a rate directly proportional to the volume of water in the container.

(a) Defining any variables, write down a differential equation that describes how the volume of water in the container varies with time.

[2]

(b) By separating the variables, find the general solution to your differential equation from part (a).

[3]

$\frac{dV}{dt}$ is rate of change of volume with respect to time

a) $\frac{dV}{dt} \propto V$

$$\boxed{\frac{dV}{dt} = -kV}$$

where V is volume, t is time, and $k>0$ is the constant of proportionality

Along with $k>0$, the minus sign here shows that the container is <u>leaking</u> — i.e., $\frac{dV}{dt} < 0$

Q4b

From part (a),

$$\frac{dV}{dt} = -kV$$

Separation of Variables

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

b) $\frac{dV}{dt} = -kV$

$$\int \frac{1}{V}\,dV = \int -k\,dt$$

$$\ln|V| = -kt + c$$

$$|V| = e^{-kt+c}$$

But V is a volume, so $V \geq 0$ and $|V| = V$

$$V = e^{-kt+c} = (e^{-kt})(e^{c})$$

$$\boxed{V = Ae^{-kt} \quad (\text{where } A = e^{c})}$$

Q5a

(a) Find the general solution to the differential equation

$$\frac{2y-1}{3}\frac{dy}{dx} = x^2y^2 - x^2y, \quad y > 1$$

[4]

(b) Find the general solution to the differential equation

$$3\frac{dy}{dx} = \frac{\operatorname{cosec} y^3}{y^2}$$

giving your answer in the form $x = f(y)$.

[4]

Separation of Variables

$$g(y)\frac{dy}{dx} = f(x) \implies \int g(y)\,dy = \int f(x)\,dx$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$ [reverse chain rule, special case]

a) $\frac{2y-1}{3}\frac{dy}{dx} = x^2y^2 - x^2y$

$\frac{2y-1}{3}\frac{dy}{dx} = x^2(y^2-y)$

$\frac{2y-1}{y^2-y}\frac{dy}{dx} = 3x^2$

$\int \frac{2y-1}{y^2-y}\,dy = \int 3x^2\,dx$

$\ln|y^2-y| = x^3 + c$

$|y^2-y| = e^{x^3+c}$ take exponential of both sides

But when $y>1$, $y^2-y = y(y-1) > 0$ so $|y^2-y| = y^2-y$

$y^2 - y = e^{x^3+c} = (e^{x^3})(e^c)$

$\boxed{y^2 - y = Ae^{x^3} \quad (\text{where } A = e^c)}$

Q5b

(a) Find the general solution to the differential equation

$$\frac{2y-1}{3}\frac{dy}{dx} = x^2y^2 - x^2y, \quad y > 1$$

[4]

(b) Find the general solution to the differential equation

$$3\frac{dy}{dx} = \frac{\operatorname{cosec} y^3}{y^2}$$

giving your answer in the form $x = f(y)$.

[4

Be careful! $\operatorname{cosec} y^3 = \operatorname{cosec}(y^3)$

It is NOT $\operatorname{cosec}^3 y = (\operatorname{cosec} y)^3$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \implies \frac{1}{\operatorname{cosec}\theta} = \sin\theta$

$\frac{d}{dy}(-\cos y^3) = 3y^2\sin y^3 \implies \int 3y^2 \sin y^3\,dy = -\cos y^3 + c$ (reverse chain rule)

Separation of Variables

$$g(y)\frac{dy}{dx} = f(x) \implies \int g(y)\,dy = \int f(x)\,dx$$

b) $3\frac{dy}{dx} = \frac{\operatorname{cosec} y^3}{y^2}$

$\frac{3y^2}{\operatorname{cosec} y^3}\frac{dy}{dx} = 1$

$\int \frac{3y^2}{\operatorname{cosec} y^3}\,dy = \int dx$

$\int 3y^2 \sin y^3\,dy = \int dx$

$-\cos y^3 + c = x$

$\boxed{x = -\cos y^3 + c}$

Alternative method

Because $\frac{1}{dy/dx} = \frac{dx}{dy}$ and $x = \int \frac{dx}{dy}\,dy$, this could also be solved as follows:

$$3\frac{dy}{dx} = \frac{\operatorname{cosec} y^3}{y^2}$$

$$\frac{3y^2}{\operatorname{cosec} y^3} = \frac{1}{dy/dx}$$

$$\frac{3y^2}{\operatorname{cosec} y^3} = \frac{dx}{dy}$$

$$x = \int \frac{3y^2}{\operatorname{cosec} y^3}\,dy$$

$$x = \int 3y^2 \sin y^3\,dy$$

$$x = -\cos y^3 + c$$

Q6a

(a) Show that the general solution to the differential equation

$$y\cot x\,\frac{dy}{dx} = y^2 + 3$$

can be written in the form

$$y^2 + 3 = A\sec^2 x$$

where A is a constant.

[5]

(b) Find the particular solution to the following differential equation, using the given boundary condition

$$e^{x^2}\frac{dy}{dx} = 2x\operatorname{cosec} 3y \qquad x = 0,\ y = \frac{\pi}{3}$$

[6]

$\int \tan kx\,dx = \frac{1}{k}\ln|\sec kx|$ (standard result)

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ [reverse chain rule, special case]

$k\ln a = \ln(a^k) \qquad \ln a + \ln b = \ln(ab)$ [Laws of logarithms]

Separation of Variables

$g(y)\frac{dy}{dx} = f(x) \implies \int g(y)\,dy = \int f(x)\,dx$

a)

$y\cot x\,\frac{dy}{dx} = y^2+3 \implies \frac{y}{y^2+3}\frac{dy}{dx} = \frac{1}{\cot x} = \tan x$

$\cot x = \frac{1}{\tan x}$

$\int \frac{y}{y^2+3}\,dy = \int \tan x\,dx$

$\int \frac{2y}{y^2+3}\,dy = 2\int \tan x\,dx$

$\ln|y^2+3| = 2\ln|\sec x| + c \qquad c = \ln(e^c)$

$\ln|y^2+3| = 2\ln|\sec x| + \ln(e^c)$

$\ln|y^2+3| = \ln(|\sec x|^2) + \ln(e^c)$

$\ln|y^2+3| = \ln(e^c|\sec x|^2)$

In general $|a|^2 = a^2$

$|y^2+3| = e^c|\sec x|^2$

But $y^2+3>0$, so $|y^2+3| = y^2+3$, and $|\sec x|^2 = \sec^2 x$

$y^2+3 = e^c\sec^2 x$

$\boxed{y^2+3 = A\sec^2 x \text{ (where } A = e^c)}$

Q6b

(a) Show that the general solution to the differential equation

$$y \cot x \frac{dy}{dx} = y^2 + 3$$

can be written in the form

$$y^2 + 3 = A \sec^2 x$$

where A is a constant.

[5]

(b) Find the particular solution to the following differential equation, using the given boundary condition

$$e^{x^2} \frac{dy}{dx} = 2x \operatorname{cosec} 3y \qquad x = 0, \; y = \frac{\pi}{3}$$

[6]

$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \implies \frac{1}{\operatorname{cosec}\theta} = \sin\theta$

$\int -3\sin 3y \, dy = \cos 3y + c$
$\int -2x e^{-x^2} \, dx = e^{-x^2} + c$
} both by reverse chain rule

Separation of Variables

$g(y)\frac{dy}{dx} = f(x) \implies \int g(y)\,dy = \int f(x)\,dx$

b) $e^{x^2}\frac{dy}{dx} = 2x\operatorname{cosec}3y \implies \frac{1}{\operatorname{cosec}3y}\frac{dy}{dx} = \frac{2x}{e^{x^2}}$

$\int \frac{1}{\operatorname{cosec}3y}\,dy = \int \frac{2x}{e^{x^2}}\,dx$

$\int \sin 3y \, dy = \int 2xe^{-x^2}\,dx$

$\int -3\sin 3y\,dy = -3\int 2xe^{-x^2}\,dx$

$\int -3\sin 3y\,dy = 3\int -2xe^{-x^2}\,dx$

$\cos 3y = 3e^{-x^2} + c$ } general solution

But when $x = 0$, $y = \pi/3$, so

$\cos(\pi) = 3e^0 + c$

$-1 = 3 + c \implies c = -4$

$\boxed{\cos 3y = 3e^{-x^2} - 4}$ } particular solution

Q7a

A large weather balloon is being inflated at a rate that is inversely proportional to the square of its volume.

(a) Defining variables for the volume of the balloon (m^3) and time (seconds) write down a differential equation to describe the relationship between volume and time as the weather balloon is inflated.

[2]

(b) Given that initially the balloon may be considered to have a volume of zero, and that after 400 seconds of inflating its volume is 600 m^3, find the particular solution to your differential equation.

[6]

(c) Although it can be inflated further, the balloon is considered ready for release when its volume reaches 1250 m^3. If the balloon needs to be ready for a midday release, what is the latest time that it can start being inflated?

[2]

$\frac{dV}{dt}$ is rate of change of volume with respect to time

a) Let V be the volume in m^3, and let t be the time in seconds.

Then $\frac{dV}{dt} \propto \frac{1}{V^2}$

$\implies \boxed{\frac{dV}{dt} = \frac{k}{V^2}}$

(where $k > 0$ is the constant of proportionality)

Because the balloon is being inflated, $\frac{dV}{dt} > 0 \implies k > 0$

Q7b

A large weather balloon is being inflated at a rate that is inversely proportional to the square of its volume.

(a) Defining variables for the volume of the balloon (m^3) and time (seconds) write down a differential equation to describe the relationship between volume and time as the weather balloon is inflated.

[2]

(b) Given that initially the balloon may be considered to have a volume of zero, and that after 400 seconds of inflating its volume is 600 m^3, find the particular solution to your differential equation.

[6]

(c) Although it can be inflated further, the balloon is considered ready for release when its volume reaches 1250 m^3. If the balloon needs to be ready for a midday release, what is the latest time that it can start being inflated?

[2]

Separation of Variables

$$g(y)\frac{dy}{dx} = f(x) \implies \int g(y)\,dy = \int f(x)\,dx$$

From part (a), $\frac{dV}{dt} = \frac{k}{V^2}$

b) $\frac{dV}{dt} = \frac{k}{V^2} \implies V^2\frac{dV}{dt} = k$

$\int V^2\,dV = \int k\,dt$

$\frac{1}{3}V^3 = kt + c$

$V^3 = 3kt + 3c$

When $t = 0$, $V = 0$

$0 = 3k(0) + 3c \implies c = 0$

$V^3 = 3kt$

When $t = 400$, $V = 600$

$600^3 = 3k(400)$

$3k = \frac{600^3}{400} = 540000$

$$\boxed{V^3 = 540000t}$$

The solution could also be given as

$V = \sqrt[3]{540000t} = 30\sqrt[3]{20t}$

Q7c

A large weather balloon is being inflated at a rate that is inversely proportional to the square of its volume.

(a) Defining variables for the volume of the balloon (m^3) and time (seconds) write down a differential equation to describe the relationship between volume and time as the weather balloon is inflated.

[2]

(b) Given that initially the balloon may be considered to have a volume of zero, and that after 400 seconds of inflating its volume is 600 m^3, find the particular solution to your differential equation.

[6]

(c) Although it can be inflated further, the balloon is considered ready for release when its volume reaches 1250 m^3. If the balloon needs to be ready for a midday release, what is the latest time that it can start being inflated?

[2]

1 hour = 3600 seconds

From part (b), $V^3 = 540000t$

c) $1250^3 = 540000t$

$t = \frac{1250^3}{540000} = 3617$ seconds (to the nearest second)

$t = 1$ hour and 17 seconds.

The balloon needs to start being inflated by about 10:59 am (and certainly no later than 17 seconds before 11 am)

Q8a

A bar of soap in the shape of a cuboid is placed in a bowl of warm water and its volume is recorded at regular intervals. The water is maintained at a constant temperature.

Before being placed in the water the soap measures 3 cm by 6 cm by 10 cm. Two minutes later the bar of soap measures 2.85 cm by 5.7 cm by 9.5 cm.

The rate of decrease in volume of the bar of soap is modelled as being directly proportional to its volume.

(a) Defining any variables you use, find and solve a differential equation linking the volume of the bar of soap and time. [7]

(b) What happens to the volume of the bar of soap for large values of t? Briefly explain why this could be considered a criticism of the model. [2]

Separation of Variables

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Note: If t is defined as time in seconds, then the solution becomes

$$\boxed{V = 180e^{-0.00128t}}$$

$\frac{dV}{dt}$ is rate of change of volume with respect to time

a) Let V = volume (cm³) and t = time (minutes).

$\frac{dV}{dt} \propto V \Rightarrow \frac{dV}{dt} = -kV$ (where $k>0$ is the constant of proportionality)

$\int \frac{1}{V}\,dV = \int -k\,dt$

$\ln|V| = -kt + c$

$|V| = e^{-kt+c} = (e^{-kt})(e^{c}) = Ae^{-kt}$ $(A = e^{c})$

But V is a volume, so $V \geq 0$ and $|V| = V$

$V = Ae^{-kt}$ } general solution

At $t = 0$, $V = 3\times6\times10 = 180$

$180 = Ae^{0} \Rightarrow A = 180$

At $t = 2$, $V = 2.85\times5.7\times9.5 = 154.3275$

$154.3275 = 180e^{-2k}$

$e^{-2k} = \frac{154.3275}{180} \Rightarrow -2k = \ln\left(\frac{154.3275}{180}\right)$

$k = -\frac{1}{2}\ln\left(\frac{154.3275}{180}\right) = 0.0769$ (3 s.f.)

$$\boxed{V = 180e^{-0.0769t}}$$ } particular solution (t in minutes)

Q8b

From part (a), $V = 180e^{-0.0769t}$

b) $e^{-0.0769t}$ is a decreasing exponential, so it gets closer and closer to zero as t gets large.

However, according to the model, the volume never reaches zero (because $180e^{-0.0769t} > 0$ for all values of t), and therefore the soap never completely dissolves.